International Journal of Theoretical Physics, Vol. 36, No. 8, 1997

Induced Electronic Interactions in Chern–Simons Systems

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Received April 11, 1997

The induced electronic interactions in (1 + 2)-dimensional vector Chern–Simons systems are studied by means of path-integral quantization. We consider four cases: relativistic, and nonrelativistic fermion Maxwell–Chern–Simons models, and relativistic and nonrelativistic fermion Chern–Simons models. It is shown that the Chern–Simons term may induce exotic electronic interactions which can be local or nonlocal and small Chern–Simons coupling may have a considerable effect in some cases.

1. INTRODUCTION

Quantum field theories in 1 + 2 dimensions involving charged matter fields minimally coupled to a Chern–Simons field exhibit anyonic sectors with exotic spin and statistics (Hagen, 1985a; Polyakov, 1988; Semenoff, 1988). Interest in such theories has been strongly stimulated by the fact that they can realize Wilczeck's charge-flux composite model, which can give a natural explanation of the fractional quantum Hall effect and its possible role in high-T_c superconductivity (Fradkin, 1991). Furthermore, they have also been extensively studied in terms of topologically massive gauge theory with the Maxwell term (Deser *et al.*, 1982). For the Maxwell–Chern–Simons model, it is clear that such a system is exactly identical to a free scalar field (Deser *et al.*, 1982; Feng and Qiu, 1955). For the Proca–Chern–Simons

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0020-7748/97/0800-1717$12.50/0

system, it has also been shown that such a system is equivalent to a system of two free scalar fields with different masses (Feng *et al.*, n.d.). Though the cases in which matter fields are also contained are much more difficult to study exactly, Jackiw and Pi (1990) and Barashenkov and Harrin (1994) were able to solve the quantum mechanical Schrödinger equations and obtain solitary solutions. The cases of fermions coupled to the Chern–Simons field have also been studied (Lopez and Fradkin, 1991). To the knowledge of the authors, the induced interactions of fermions due to the coupling to the Chern–Simons field has not been discussed; this the subject of this paper. We will study four cases. In Section 2, we study the relativistic fermion Chern–Simons systems with and without the Maxwell term. In Section 3, we study the nonrelativistic fermion Chern–Simons systems with and without the Maxwell term. Our approach is the path-integral quantization for constrained systems. Some notations are only applicable in a local context and this should not lead to any confusion, while other notations, if not otherwise defined, are the same as usually used.

2. RELATIVISTIC CHERN–SIMONS MODELS

2.1. Fermion Maxwell–Chern–Simons Model

The model Lagrangian is (Deser *et al.*, 1982)

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - m\bar{\psi}\psi - \frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4} \epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \tag{1}$$

where $\not{D} = \gamma^\mu D_\mu$, $D_\mu = \partial_\mu + ieA_\mu$, $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$, $\gamma^2 = i\sigma_1$, and σ_i ($i = 1, 2, 3$) are the Pauli matrices. The Euler–Lagrangian equations are

$$(i\not{D} - m)\psi = 0 \tag{2}$$

$$\partial_\nu F^{\nu\mu} + \frac{\mu}{2} \epsilon^{\mu\nu\rho}F_{\nu\rho} = eJ^\mu, \qquad J^\mu = \bar{\psi}\gamma^\mu\psi \tag{3}$$

In the Hamiltonian description, the canonical momenta are

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\bar{\psi}\gamma^0, \qquad \pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = F^{\mu 0} + \frac{\mu}{2} \epsilon^{0\mu\nu}A_\nu \tag{4}$$

The canonical Hamiltonian is

$$\begin{aligned}\mathcal{H}_c = \pi\dot{\psi} + \pi^\mu\dot{A}_\mu - \mathcal{L} = eA_0\bar{\psi}\gamma^0\psi - i\bar{\psi}\gamma^i D_i\psi + m\bar{\psi}\psi - \frac{1}{2}\pi^i\pi_i \\ + \frac{\mu}{2}\epsilon^{ij}\pi_i A_j - \frac{\mu^2}{8}A^i A_i + \frac{1}{4}F^{ij}F_{ij} \\ - \frac{\mu}{4}\epsilon^{ij}F_{ij}A_0 + \pi^i\partial_i A_0\end{aligned} \tag{5}$$

From equation (4) we have a primary constraint

$$\chi_1 = \pi^0 \approx 0 \tag{6}$$

The total Hamiltonian is then

$$\mathcal{H}_T = \mathcal{H}_c + \alpha\chi_1 \tag{7}$$

From the consistency condition

$$\{\chi_1, H_T\} \approx 0, \qquad H_T = \int d^2\mathbf{x}\, \mathcal{H}_c \tag{8}$$

we have a secondary constraint

$$\chi_2 = \frac{\pi}{4}\,\epsilon^{ij}F_{ij} + \partial_i\pi^i - e\bar{\psi}\gamma^0\psi \approx 0 \tag{9}$$

It is easy to show that

$$\{\chi_2, \mathcal{H}_T\} = 0, \qquad \{\chi_1, \chi_2\} = 0 \tag{10}$$

so χ_1, χ_2 are first-class constraint. According to the standard procedure (Gitmann and Tyutin, 1990), we should choose two gauge-fixing conditions

$$f_1 = \partial^i A_i \approx 0, \qquad f_2 = \partial_i\pi^i + \nabla^2 A_0 - \frac{\mu}{2}\,\epsilon^{ij}\partial_i A_j \tag{11}$$

where $\nabla^2 = -\partial^i\partial_i$. Since det $\{\chi_i, f_i\}$ = const, the quantum generating functional is

$$\begin{aligned} Z[\bar{\eta}, \eta, j^\mu] = \int &\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\pi^\mu\,\mathcal{D}A_\mu\,\delta(\chi_i)\delta(f_i) \\ &\times \exp\left\{ i \int d^3x\,[i\bar{\psi}\gamma^0\dot{\psi} - \pi^\mu\dot{A}_\mu - \mathcal{H}_c \right. \\ &\left. + \bar{\eta}\psi + \bar{\psi}\eta + j^\mu A_\mu] \right\} \end{aligned} \tag{12}$$

π^0 can be integrated out readily,

$$\begin{aligned} Z[\bar{\eta}, \eta, j^\mu] = \int &\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A_\mu\,\mathcal{D}\pi^i\,\delta(\chi_2)\delta(f_i) \\ &\times \exp\left\{ i \int d^3x\,[i\bar{\psi}\gamma^0\dot{\psi} + \pi^i A_i - \mathcal{H}_c \right. \\ &\left. + \bar{\eta}\psi + \bar{\psi}\eta + j^\mu A_\mu] \right\} \end{aligned} \tag{13}$$

Making a displacement $\pi^i \to \pi^i - (\mu/2)\epsilon^{ij}A_j$, we have

$$
\begin{aligned}
Z[\bar{\eta}, \eta, j^\mu] = \int & \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}A_\mu\, \mathcal{D}\pi^i\, \delta(\partial_i\pi^i - e\bar{\psi}\gamma^0\psi) \\
& \times \delta(\partial^i A_i)\delta(\nabla^2 - \mu\epsilon^{ij}\partial_i A_j + \partial_i\pi^i) \\
& \times \exp\left\{ i \int d^3x \left[\bar{\psi} i \not{D} \psi - m\bar{\psi}\psi + \bar{\psi}\eta + \bar{\eta}\psi \right.\right. \\
& + j^\mu A_\mu + \pi^i \dot{A}_i - \frac{\mu}{2}\epsilon^{ij}\dot{A}_i A_j + \frac{1}{2}\pi^i\pi_i + \frac{\mu^2}{2}A^i A_i - \frac{1}{4}F^{ij}F_{ij} \\
& \left.\left. - \mu\epsilon_{ij}\pi^i A^j - \pi^i\partial_i A_0 \right]\right\}
\end{aligned}
\tag{14}
$$

Since the three delta functionals ensure that

$$
A_0 \approx \frac{1}{\nabla^2}(\mu\epsilon^{ij}\partial_i A_j - \partial_i\pi^i) \tag{15}
$$

and

$$
\partial_k A_0 \approx -\mu\epsilon_{ki}A^i - \frac{\partial_k}{\nabla^2} e\bar{\psi}\gamma^0\psi \tag{16}
$$

we have

$$
-\pi^i\partial_i A_0 \approx \mu\epsilon_{ij}\pi^i A^j - e^2\bar{\psi}\gamma^0\psi\frac{1}{\nabla^2}(\bar{\psi}\gamma^0\psi) \tag{17}
$$

so

$$
\begin{aligned}
Z[\bar{\eta}, \eta, j^\mu] = \int & \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}A_\mu\, \mathcal{D}\pi^i\, \delta(\partial_i\pi^i - e\bar{\psi}\gamma^0\psi) \\
& \times \delta(\partial^i A_i)\delta(\nabla^2 - \mu\epsilon^{ij}\partial_i A_j + \partial_i\pi^i) \\
& \times \exp\left\{ i \int d^3x \left[\bar{\psi} i \not{D} \psi - m\bar{\psi}\psi + \bar{\psi}\eta + \bar{\eta}\psi \right.\right. \\
& + j^\mu A_\mu + \pi^i \dot{A}_i - \frac{\mu}{2}\epsilon^{ij}\dot{A}_i A_j + \frac{1}{2}\pi^i\pi_i + \frac{\mu^2}{2}A^i A_i \\
& \left. - \frac{1}{4}F^{ij}F_{ij} - e^2\bar{\psi}\gamma^0\psi\frac{1}{\nabla^2}(\bar{\psi}\gamma^0\psi) \right\}
\end{aligned}
\tag{18}
$$

Because of equation (16), we have

$$\mu\epsilon^{ij}\dot{A}_i A_j \approx \partial_i \left[A^i\left(A_0 + \frac{1}{\nabla^2} e\bar{\psi}\gamma^0\psi\right)\right] = \text{surface term} \tag{19}$$

so it is negligible. After integrating A_0, we have

$$\begin{aligned} Z[\bar{\eta}, \eta] &\equiv Z[\bar{\eta}, \eta, j^\mu]_{j^\mu=0} \\ &= \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\pi^i\,\mathcal{D}A_i\,\delta(\partial_i\pi^i - e\bar{\psi}\gamma^0\psi)\delta(\partial^i A_i) \\ &\quad\times \exp\left\{i\int d^3x\left[\bar{\psi}i\not{D}\psi - m\bar{\psi}\psi + \bar{\psi}\eta + \bar{\eta}\psi + \pi^i\dot{A}_i + \frac{1}{2}\pi^i\pi_i\right.\right. \\ &\quad\left.\left. + \frac{\mu^2}{2}A^iA_i - \frac{1}{4}F_{ij}F^{ij} - e\bar{\psi}\gamma^0\psi\frac{\mu}{\nabla^2}\epsilon^{ij}\partial_i A_j\right]\right\} \end{aligned} \tag{20}$$

Decompose π^i as

$$\pi_i = \frac{\epsilon^{ij}\partial_j\phi}{\sqrt{-\nabla^2}} + \partial^i v, \qquad \partial^i\partial_i v = e\bar{\psi}\gamma^0\psi \tag{21}$$

Then $\mathcal{D}\pi^i\delta(\partial_i\pi^i - eJ^0) = \text{const}\cdot\mathcal{D}\phi$ and

$$\pi^i\dot{A}_i + \frac{1}{2}\pi^iA_i \approx -\phi\frac{\epsilon^{ij}\partial_j\dot{A}_i}{\sqrt{-\nabla^2}} - \frac{1}{2}\phi^2 - \frac{1}{2}v\partial^i\partial_i v \tag{22}$$

Since

$$\begin{aligned} &\int \mathcal{D}\phi\,\exp\left\{i\int d^3x\left(-\phi\frac{\epsilon^{ij}\partial_j\dot{A}_i}{\sqrt{-\nabla^2}} - \frac{1}{2}\phi^2\right)\right\} \\ &\quad = \exp\left\{i\int d^3x\left(-\frac{1}{2}\dot{A}^i\dot{A}_i\right)\right\} \end{aligned} \tag{23}$$

we have

$$\begin{aligned} Z[\bar{\eta}, \eta] = &\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A_i\,\delta(\partial^iA_i)\exp\left\{i\int d^3x\left[\bar{\psi}i\not{\partial}\psi - m\bar{\psi}\psi - e\bar{\psi}\gamma^i\psi A_i\right.\right. \\ &+ \bar{\psi}\eta + \bar{\eta}\psi - \frac{1}{2}\dot{A}^i\dot{A}_i + \frac{\mu^2}{2}A^iA_i - \frac{1}{4}F^{ij}F_{ij} \\ &\left.\left. - e\bar{\psi}\gamma^0\psi\frac{\mu}{\nabla^2}\epsilon^{ij}\partial_iA_j + \frac{1}{2}J^0\frac{1}{\nabla^2}J^0\right]\right\} \end{aligned} \tag{24}$$

Due to $\delta(\partial^i A_i)$, we can write $A_i = (\epsilon_{ij}\partial^j/\sqrt{-\nabla^2})\varphi$ and $\mathcal{D}A_i\delta(\partial^i A_i) =$ const$\cdot\mathcal{D}\varphi$, so

$$-\frac{1}{2}\dot{A}^i\dot{A}_i + \frac{\mu^2}{2}A^iA_i - \frac{1}{4}F^{ij}F_{ij} = \frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi - \mu^2\varphi^2) \tag{25}$$

and

$$\begin{aligned} Z[\bar{\eta}, \eta] = & \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\varphi \exp\left\{i\int d^3x\left[\bar{\psi}i\partial\!\!\!/\psi - m\bar{\psi}\psi - e\bar{\psi}\gamma^i\psi A_i + \bar{\psi}\eta\right.\right. \\ & + \bar{\eta}\psi - eJ^0\frac{\mu}{\sqrt{-\nabla^2}}\varphi + \frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi - \mu^2\varphi^2) \\ & \left.\left. + \frac{\epsilon_{ij}\partial^jJ^i}{\sqrt{-\nabla^2}}\varphi + \frac{1}{2}J^0\frac{1}{\nabla^2}J^0\right]\right\} \\ = & \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left\{i\int d^3x\left[\bar{\psi}i\partial\!\!\!/\psi - m\bar{\psi}\psi + \bar{\psi}\eta + \bar{\eta}\psi\right.\right. \\ & \left.\left. - \frac{1}{2}J^0\frac{\mu^2}{\nabla^2}DJ^0 + \frac{e^2}{2}\frac{\epsilon_{ij}\partial^jJ^i}{\sqrt{-\nabla^2}}D\frac{\epsilon_{kl}\partial^lJ^k}{\sqrt{-\nabla^2}} + \frac{1}{2}J_0\frac{1}{\nabla^2}J^0\right]\right\} \end{aligned} \tag{26}$$

where $D^{-1} = \partial^\mu\partial_\mu + \mu^2$, i.e.,

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left\{i\int d^3x\,\mathcal{L}_{\text{eff}} + \bar{\eta}\psi + \bar{\psi}\eta\right\} \tag{27}$$

where

$$\mathcal{L}_{\text{eff}} = \bar{\psi}i\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{e^2}{2}J^\mu DJ_\mu \tag{28}$$

This is the effective Lagrangian. The interaction induced by the vector field A_μ is then $\mathcal{H}_{\text{int}} = 1/2\, e^2J^\mu DJ_\mu$, which is nonlocal.

2.2. The Fermion Chern–Simons Model

The model Lagrangian is

$$\mathcal{L} = \bar{\psi}iD\!\!\!\!/\,\psi - m\bar{\psi}\psi + \frac{\mu}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \tag{29}$$

The canonical momenta are given by definition as

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\bar{\psi}\gamma^0, \qquad \pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = \frac{\mu}{2}\epsilon^{0\mu\nu}A_\nu \tag{30}$$

and the canonical Hamiltonian is

$$\mathcal{H}_c = -i\bar{\psi}\gamma^i D_i\psi + m\bar{\psi}\psi + e\bar{\psi}\gamma^0\psi A_0 - \mu\epsilon^{ij}\partial_i A_j A_0 \tag{31}$$

We have the primary constraints

$$\chi_1 = \pi^0 \approx 0, \qquad C^i = \pi^i - \frac{\mu}{2}\epsilon^{ij}A_j \approx 0 \tag{32}$$

with the Poisson brackets

$$\{\chi_1, C^i\} = 0, \qquad \{C^i(\mathbf{x}), C^j(\mathbf{y})\} = -\mu\epsilon^{ij}\delta(\mathbf{x} - \mathbf{y}) \tag{33}$$

The total Hamiltonian is thus

$$\mathcal{H}_T = \mathcal{H}_c + \lambda\chi_1 + \lambda_i C^i \tag{34}$$

Since

$$\left\{\chi_1, \int d^2\mathbf{x}\; \mathcal{H}_T\right\} = -e\bar{\psi}\gamma^0\psi + \mu\epsilon^{ij}\partial_i A_j \tag{35}$$

$$\left\{C^i, \int d^2\mathbf{x}\; \mathcal{H}_T\right\} = \left\{C^i, \int d^2\mathbf{x}\; \mathcal{H}_c\right\} + \lambda_j\left\{C^i, \int d^2\mathbf{x}\; C^j\right\} \tag{36}$$

we have a secondary constraint

$$\xi = \mu\epsilon^{ij}\partial_i A_j - e\bar{\psi}\gamma^0\psi \approx 0 \tag{37}$$

with

$$\{\xi, \chi_1\} = 0, \qquad \{\xi(\mathbf{x}), C^j(\mathbf{y})\} = \mu\epsilon^{ij}\partial_i^x\, \delta(\mathbf{x} - \mathbf{y}) \tag{38}$$

Equation (36) does not lead to any new constraint, whereas it fixes the Lagrange multipliers λ_j. As in Kim *et al.* (1995) and Ni and Chen (1995), in order to extract the true second class constraint, it is essential to define

$$\chi_2 = \xi + \partial_i C^i \tag{39}$$

Then

$$\{\chi_2, \chi_1\} = 0, \qquad \{\chi_2, C^j\} = 0 \tag{40}$$

So we have two first-class constraints χ_i and two second-class constraints C^i, $i = 1, 2$. Correspondingly, two gauge conditions have to be chosen. They should be such that no contradictions will appear. A possible appropriate choice is

$$f_1 = \nabla^2 A^0 - \frac{e}{\mu}\epsilon_{ij}\partial^i J^j, \qquad f_2 = \epsilon_{ij}\partial^i\pi^j \tag{41}$$

where J^μ is also given by equation (3). It can be shown that $\det\{f_i, \chi_j\}$ = const, hence we have the generating functional

$$
\begin{aligned}
Z[\bar{\eta}, \eta] = \int & \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}\pi^\mu\, \mathcal{D}A_\mu\, \delta(\chi_i)\delta(C^i)\delta(f_i) \\
& \times \exp\left\{i \int d^3x\, [i\bar{\psi}\gamma^0\dot{\psi} + \pi^\mu \dot{A}_\mu \right. \\
& \left. - e\bar{\psi}\gamma^0\psi A_0 + \bar{\psi}i\gamma^i D_i\psi - m\bar{\psi}\psi + \bar{\psi}\eta + \bar{\eta}\psi]\right\} \\
= \int & \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}A_i\, \delta(\xi)\delta(\partial_i A^i) \\
& \times \exp\left\{i \int d^3x \left[i\bar{\psi}\gamma^0\dot{\psi} \right.\right. \\
& \left.\left. + \frac{\mu}{2}\epsilon^{ij}A_j\dot{A}_i + \bar{\psi}_i\gamma^i D_i - m\bar{\psi}\psi + \bar{\psi}\eta + \bar{\eta}\psi\right]\right\}
\end{aligned}
\tag{42}
$$

Let $A_i = \partial_i\varphi$; we have

$$
\begin{aligned}
Z[\bar{\eta}, \eta] = \int & \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}\varphi\, \delta\left(\varphi - \frac{e}{\mu}\frac{1}{\nabla^2}J^0\right) \\
& \times \exp\left\{i \int d^3x \left[\bar{\psi}i\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{\mu}{2}\epsilon_{ij}\partial^i\varphi\partial^j\varphi \right.\right. \\
& \left.\left. - e\bar{\psi}\gamma^i\psi\epsilon_{ij}\partial^j\varphi + \bar{\eta}\psi + \bar{\psi}\eta\right]\right\} \\
= \int & \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \exp\left\{i \int d^3x\, (\mathcal{L}_{\text{eff}} + \bar{\eta}\psi + \bar{\psi}\eta\right\}
\end{aligned}
\tag{43}
$$

where

$$
\mathcal{L}_{\text{eff}} = \bar{\psi}i\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{e^2}{\mu}\bar{\psi}\gamma^i\psi\frac{\epsilon_{ij}\partial^j}{\nabla^2}(\bar{\psi}\gamma^0\psi) \tag{44}
$$

It can be seen that the interaction induced by the Chern–Simons coupling is proportional to the inverse of μ, so small μ may have considerable effects. From $\nabla^{-2} \sim \ln r$, we know that the potential decreases as $1/r$, like the Coulomb force, in 1 + 3 dimensions. Furthermore, $\mathcal{L}_{\text{eff}}$ is equivalent to the direct substitution of the solution of A_μ given in (Jackiw and Pi, 1990; Hagen, 1985b) into the original Lagrangian, so it is actually covariant.

3. NONRELATIVISTIC CHERN–SIMONS MODELS

3.1. Fermion Maxwell–Chern–Simons Model

The model Lagrangian is supposed to be

$$\mathcal{L} = \frac{1}{2}[\psi^\dagger iD_0\psi + (iD_0\psi)^\dagger\psi] + \frac{1}{2m}(\not{D}\psi)^\dagger\not{D}\psi - \frac{\theta}{4}F^{\mu\nu}F_{\mu\nu}$$
$$+ \frac{\mu}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \tag{45}$$

where $\not{D} = \gamma^i D_i$. Aside from a surface term, it is equivalent to

$$\mathcal{L} = \psi^\dagger iD_0\psi - \frac{1}{2m}\psi^\dagger\not{D}^2\psi - \frac{\theta}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \tag{46}$$

where $\not{D}^2 = D_iD^i + 1/4[\gamma^i, \gamma^j]ieF_{ij}$. The Euler–Lagrange equation for ψ is

$$iD_0\psi = \frac{1}{2m}\not{D}^2\psi \tag{47}$$

and those for A_μ are

$$\theta\partial_\mu F^{\mu\nu} + \frac{\mu}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = eJ^\nu \tag{48}$$

$$J^\nu = \left[\psi^\dagger\psi, \frac{i}{2m}(\psi^\dagger D^i\psi - (D^i\psi)^\dagger\psi) - \frac{1}{2m}\epsilon^{ij}\partial_j(\psi^\dagger\sigma_3\psi)\right] \tag{49}$$

The canonical momenta are

$$\pi = \frac{\partial\mathcal{L}}{\partial\psi} = i\psi^\dagger, \qquad \pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = \theta F^{\mu 0} + \frac{\mu}{2}\epsilon^{\mu ij}A_j \tag{50}$$

and the canonical Hamiltonian is

$$\mathcal{H}_c = \frac{1}{2m}\psi^\dagger\not{D}^2\psi + e\psi^\dagger\psi A_0 - \frac{1}{2}\pi^i\pi_i + \frac{\mu}{2}\epsilon^{ij}\pi_i A_j - \frac{\mu^2}{8}A^iA_i$$
$$+ \frac{\theta}{4}F^{ij}F_{ij} - \frac{\mu}{4}\epsilon^{ij}F_{ij}A_0 + \pi^i\partial_i A_0 \tag{51}$$

We have a primary constraint $\chi_1 \approx 0$. The total Hamiltonian is

$$H_T = \int (\mathcal{H}_c + \alpha\chi_1) d^3x \tag{52}$$

Conservation of $\chi_1 = 0$ leads to a secondary constraint

$$\chi_2 = e\psi^\dagger\psi - \frac{\mu}{4}\epsilon^{ij}F_{ij} - \partial_i\pi^i \approx 0 \tag{53}$$

There are no further constraints. It can be verified that the χ_i are first class and therefore we should choose two gauge conditions. They can be chosen as

$$f_1 = \partial^i A_i, \qquad f_2 = \partial_i\pi^i + \nabla^2 A_0 - \frac{\mu}{2}\epsilon^{ij}\partial_i A_j \tag{54}$$

Since $\det\{\chi_i, f_j\} = \text{const}$, we have the quantal generating functional

$$Z[\eta^\dagger, \eta] = \int \mathcal{D}\psi^\dagger\, \mathcal{D}\psi\, \mathcal{D}\pi^\mu\, \mathcal{D}A_\mu\, \delta(\chi_i)\delta(f_i)$$
$$\times \exp\left\{ i\int d^3x[\pi\dot\psi + \pi^\mu \dot A_\mu - \mathcal{H}_c + \eta^\dagger\psi + \psi^\dagger\eta] \right\} \tag{55}$$

Making the translation $\pi^i \to \pi^i - \frac{1}{2}\mu\, \epsilon^{ij}A_j$, we have

$$Z[\eta^\dagger, \eta] = \int \mathcal{D}\psi^\dagger\, \mathcal{D}\psi\, \mathcal{D}\pi^i\, \mathcal{D}A_\mu\, \delta(\partial^i A_i)\delta(\partial_i\pi^i - e\psi^\dagger\psi)$$
$$\times\, \delta(\partial_i\pi^i - \mu\epsilon^{ij}\partial_i A_j + \nabla^2 A_0)$$
$$\times \exp\left\{ i\int d^3x[\psi^\dagger iD_0\psi - \frac{1}{2m}\psi^\dagger \mathbf{D}^2\psi + \pi^i \dot A_i - \frac{\mu}{2}\epsilon^{ij}A_j\dot A_i \right.$$
$$\left. + \frac{1}{2}\pi^i\pi_i + \frac{\mu^2}{2}A^iA_i - \frac{\theta}{4}F^{ij}F_{ij} - \pi^i\partial_i A_0 + \eta^\dagger\psi + \psi^\dagger\eta] \right\} \tag{56}$$

Decomposing $\pi^i = (\epsilon^{ij}\partial_j/\sqrt{-\nabla^2})\phi + \partial^i v$, $\partial^i\partial_i v = e\psi^\dagger\psi$, we have

$$Z[\eta^\dagger, \eta] = \int \mathcal{D}\psi^\dagger\, \mathcal{D}\psi\, \mathcal{D}\phi\, \mathcal{D}A_\mu\, \delta(\partial^i A_i)\delta(-\mu\epsilon^{ij}\partial_i A_j + \nabla^2 A_0 + e\psi^\dagger\psi)$$
$$\times \exp\left\{ i\int d^3x[-\phi\frac{\epsilon^{ij}\partial_j}{\sqrt{-\nabla^2}}\dot A_i - \frac{1}{2}\phi^2 - \frac{1}{2}v\partial^i\partial_i v\psi^\dagger iD_0\psi \right.$$
$$- \frac{1}{2m}\psi^\dagger\mathbf{D}^2\psi - \frac{\mu}{2}\epsilon^{ij}A_j\dot A_i + \frac{\mu^2}{2}A^iA_i$$

$$- \frac{\theta}{4} F^{ij}F_{ij} + e\psi^\dagger\psi A_0 + \eta^\dagger\psi + \psi^\dagger\eta]\bigg\}$$

$$= \int \mathcal{D}\psi^\dagger\, \mathcal{D}\psi\, \mathcal{D}A_\mu\, \delta(\partial^i A_i)\delta(-\mu\epsilon^{ij}\partial_i A_j + \nabla^2 A_0 + e\psi^\dagger\psi)$$

$$\times \exp\bigg\{i \int d^3x[\psi^\dagger i\dot{\psi} - \frac{1}{2} \nu\partial^i\partial_i\nu - \frac{1}{2m} \psi^\dagger \mathbf{D}^2\psi - \frac{1}{2} \dot{A}^i\dot{A}_i$$

$$- \frac{\mu}{2} \epsilon^{ij}A_j\dot{A}_i + \frac{\mu^2}{2} A^iA_i - \frac{\theta}{4} F^{ij}F_{ij} + \eta^\dagger\psi + \psi^\dagger\eta]\bigg\} \tag{57}$$

Since from equation (19), $-\mu\epsilon^{ij}\dot{A}_iA_j \approx$ surface term, we have

$$Z[\eta^\dagger, \eta] = \int \mathcal{D}\psi^\dagger\, \mathcal{D}\psi\, \mathcal{D}A_i\, \delta(\partial^i A_i) \exp\bigg\{i \int d^3x[\psi^\dagger i\dot{\psi} - \frac{1}{2} \nu\partial^i\partial_i\nu - \frac{1}{2m} \psi^\dagger\mathbf{D}^2\psi$$

$$- \frac{1}{2} \dot{A}^i\dot{A}_i + \frac{\mu^2}{2} A^iA_i - \frac{\theta}{4} F^{ij}F_{ij} + \eta^\dagger\psi + \psi^\dagger\eta]\bigg\} \tag{58}$$

Let $A_i = (\epsilon_{ij}\partial^j/\sqrt{-\nabla^2})\varphi$; we have

$$Z[\eta^\dagger, \eta] = \int \mathcal{D}\psi^\dagger\, \mathcal{D}\psi\, \mathcal{D}\varphi \exp\bigg\{i \int d^3x(\mathcal{L}_{\text{eff}} + \eta^\dagger\psi + \psi^\dagger\eta)\bigg\} \tag{59}$$

where

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} (\dot{\varphi}^2 + \theta\partial^i\varphi\partial_i\varphi - \mu^2\varphi^2) + \psi^\dagger i\dot{\psi} + \frac{1}{2m} \psi^\dagger\nabla^2\psi + \frac{1}{2} \nu\nabla^2\nu$$

$$- \frac{ie}{m} \psi^\dagger\partial_i\psi \frac{\epsilon^{ij}\partial_j}{\sqrt{-\nabla^2}} \varphi - \frac{e^2}{2m} \psi^\dagger\psi\varphi^2 - \frac{e}{2m} \psi^\dagger\sigma_3\psi \sqrt{-\nabla^2}\varphi \tag{60}$$

The new field φ cannot be integrated exactly. Hence, for such a system, the induced interactions can only be obtained approximately, unlike in the relativistic case. {Note that the mass dimensions in (1 + 2)-dimensional spacetime are: $[\mu] = [m]$, $[\varphi] = [m]^{1/2}$, $[\psi] = [m]$, $[e] = [m]^{1/2}$, $[A_\mu] = [m]^{1/2}$}

3.2. Nonrelativistic Fermion Chern–Simons Model

The model Lagrangian is

$$\mathcal{L} = \psi^\dagger iD_0\psi - \frac{1}{2m} \psi^\dagger\mathbf{D}^2\psi + \frac{\mu}{4} \epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \tag{61}$$

where

$$\psi^\dagger \mathbf{D}^2\psi = \psi^\dagger(-\nabla^2)\psi + \psi^\dagger\psi ie\partial^i A_i + 2ie\psi^\dagger\partial^i\psi A_i - e^2\psi^\dagger\psi A^i A_i + \frac{ie}{4}\psi^\dagger[\gamma^i, \gamma^j]\psi F_{ij} \tag{62}$$

The canonical momenta are

$$\pi = \frac{\partial\mathcal{L}}{\partial\psi} = i\psi^\dagger, \qquad \pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = \frac{\mu}{2}\epsilon^{0\mu\nu}A_\nu \tag{63}$$

Hence we have the primary constraints

$$\chi_1 = \pi^0 \approx 0, \qquad C^i = \pi^i - \frac{\mu}{2}A_j \approx 0 \tag{64}$$

with Poisson brackets

$$\{\chi_i, C^i\} = 0, \qquad \{C^i(\mathbf{x}), C^j(\mathbf{y})\} = -\mu\epsilon^{ij}(\mathbf{x}-\mathbf{y}) \tag{65}$$

The canonical and total Hamiltonians are therefore

$$\mathcal{H}_c = \frac{1}{2m}\psi^\dagger\mathbf{D}^2\psi + e\psi^\dagger\psi A_0 - \mu\epsilon^{ij}\partial_i A_j A_0 \tag{66}$$

$$\mathcal{H}_T = \mathcal{H}_c + \lambda\chi_1 + \lambda_i C^i \tag{67}$$

From the consistency condition $\{\chi_1, \int d^2\mathbf{x}\mathcal{H}_T\} \approx 0$, we have a secondary constraint

$$\xi = -e\psi^\dagger\psi + \mu\epsilon^{ij}\partial_i A_j \approx 0 \tag{68}$$

with

$$\{\xi, \mathcal{H}_T\} = 0, \qquad \{C^i(\mathbf{x}), \xi(\mathbf{y})\} = \mu\epsilon^{ij}\partial_j^y\delta(\mathbf{x}-\mathbf{y}) \tag{69}$$

As in the relativistic case, $\{C^i, \mathcal{H}_T\} \approx 0$ gives the determination of λ_i and leads to no new constraint. Similarly, we define

$$\chi_2 = \xi + \partial_i C^i \tag{70}$$

Then we have

$$\{\chi_1, \chi_2\} = 0, \qquad \{\chi_2, C^i\} = 0 \tag{71}$$

So χ_i are first class and C^i are second class. Correspondingly, we choose the gauge-fixing conditions as in the relativistic case,

$$f_1 = \nabla^2 A^0 - \frac{e}{\mu}\epsilon_{ij}\partial^i J^j, \qquad f_2 = \epsilon_{ij}\partial^i\pi^j \tag{72}$$

where J^i are given by equation (49). Since det $\{f_i, \chi_j\}$ = const, det$\{C^i, C^j\}$ = const, we have

$$Z[\eta\dagger, \eta] = \int \mathcal{D}\psi\dagger\, \mathcal{D}\psi\, \mathcal{D}\pi^\mu\, \mathcal{D}A_\mu\, \delta(\chi_i)\delta(C^i)\delta(f_i) \exp\left\{ i \int d^3x[i\psi\dagger\dot{\psi} + \pi^\mu \dot{A}_\mu - \frac{1}{2m}\psi\dagger \mathbf{D}^2\psi - e\psi\dagger\psi A_0 + \mu\epsilon^{ij}\partial_i A_j A_0 + \eta\dagger\psi + \psi\dagger\eta] \right\} \tag{73}$$

Since

$$\mu\epsilon_{ij}\partial^i A^j - e\psi\dagger\psi \approx 0 \quad (\text{mod } \xi) \tag{74}$$

we have

$$\begin{aligned} Z[\eta\dagger, \eta] &= \int \mathcal{D}\psi\dagger\, \mathcal{D}\psi\, \mathcal{D}\pi^i\, \mathcal{D}A_i\, \delta(\xi)\delta(C^i)\delta(f_2) \\ &\quad \times \exp\left\{ i \int d^3x[i\psi\dagger\dot{\psi} + \pi^i \dot{A}_i - \frac{1}{2m}\psi\dagger \mathbf{D}^2\psi + \eta\dagger\psi + \psi\dagger\eta] \right\} \\ &= \int \mathcal{D}\psi\dagger\, \mathcal{D}\psi\, \mathcal{D}A_i\, \delta(\xi)\delta(\partial^i A_i) \\ &\quad \times \exp\left\{ i \int d^3x[i\psi\dagger\dot{\psi} + \frac{\mu}{2}\epsilon^{ij}A_j\dot{A}_i - \frac{1}{2m}\psi\dagger \mathbf{D}^2\psi + \eta\dagger\eta + \psi^\dagger\eta] \right\} \end{aligned} \tag{75}$$

Let $A_i = \epsilon_{ij}\partial^j\varphi$; then $F_{ij} = \epsilon_{ij}\nabla^2\varphi$, φ can be integrated

$$Z[\eta\dagger, \eta] = \int \mathcal{D}\psi\dagger\, \mathcal{D}\psi \exp\left\{ i \int d^3x[\mathcal{L}_{\text{eff}} + \eta\dagger\psi + \psi\dagger\eta] \right\} \tag{76}$$

where

$$\mathcal{L}_{\text{eff}} = i\psi\dagger\dot{\psi} + \frac{1}{2m}\psi\dagger\nabla^2\psi - \frac{ie^2}{m\mu}\psi\dagger\partial^i\psi\epsilon_{ij}\frac{\partial^j}{\nabla^2}(\psi\dagger\psi) + \frac{e^4}{2m\mu^2}\psi\dagger\psi\frac{\partial^i}{\nabla^2}(\psi\dagger\psi)\frac{\partial_i}{\nabla^2}(\psi\dagger\psi) - \frac{ie^2}{8m\mu}\psi\dagger[\gamma^i, \gamma^j]\psi\epsilon_{ij}\psi\dagger\psi \tag{77}$$

This effective Lagrangian involves a six-fermion interaction and a local four interaction. As in the relativistic case, $\mathscr{L}_{\text{eff}}$ equals that obtained by directly substituting the classical solutions of A_μ given in Jackiw and Pi (1990) and Hagen (1985b) into the original Lagrangian, and this is because that without the Maxwell term, the vector field has no dynamics.

4. DISCUSSION

The discussion of induced electronic interactions is similar to that from the electron–phonon Hamiltonian to Fröhlich's Hamiltonian in which the phonon degrees are integrated out (Jones and March, 1973). Since the origin of Chern–Simons coupling is becoming clearer, e.g., it may be generated by a heavy fermion determinant (Redlich, 1984a, b; Yang and Ni, 1995), we may suppose that there exist two species of fermions, ψ_m and ψ_M with masses m and M, and in the limit $M \to \infty$, the self-interaction of ψ_m induced by ψ_M is equivalent to that induced by a Chern–Simons field. The concept of a heavy fermion is not purely imaginary, it does exist as a kind of quasiparticle (Andres *et al.*, 1975; Feng and Jin, 1992).

Since the interactions are now known, we may study the properties of the electronic system. For the ground state, we may use, for example, the density functional method and the Gaussian effective potential method. On the other hand, in the discretized case for a lattice, we may study it in the Bloch representation (BCS-like) as well as the Wannier representation (Hubbard-like), for which the η-pairing method is especially powerful (Yang, 1989).

ACKNOWLEDGMENTS

S.-S.F. is indebted to Prof. G. J. Ni for his help and discussions. This work was supported in part by the National Natural Science Foundation of China.

REFERENCES

Andres, K., Graebner, J. E., and Ott, H. R. (1975). *Physical Review Letters*, **35**, 1779.

Barashenkov, I. V., and Harrin, A. O. (1994). *Physical Review Letters*, **72**, 1575.

Deser, S., Jackiw, R., and Templeton, S. (1982). *Annals of Physics*, **140**, 372.

Feng, D., and Jin, G. H. (1992). *New Perspectives on Condensed Matter Physics*, Shanghai Scientific and Technical Publishers, Shanghai.

Feng, S. S., and Qiu, X. J. (1995). *International Journal of Theoretical Physics*, **34**, 1827.

Feng, S. S., Qiu, X. J., and Zhu, Z. Y. (1997). *International Journal of Theoretical Physics*, **36**(1):41.

Fradkin, E. (1991). *Field Theories of Condensed Matter Physics*, Addison-Wesley, Reading, Massachusetts.
Gitmann, D. M., and Tyutin, I. V. (1990). *Quantization of Fields with Constraints*, Springer-Verlag, Berlin.
Hagen, C. R. (1985a). *Physical Review Letters*, **61**, 2135.
Hagen, C. R. (1985b). *Physical Review D*, **31**, 848.
Jackiw, R., and Pi, S. Y. (1990). *Physical Review Letters*, **64**, 2969.
Jones, W., and March, N. H. (1973). *Theoretical Solid State Physics*, Wiley, New York.
Kim, W. W., Park, Y. J., Kim, K. Y., and Kim, Y. (1995). *Physical Review D*, **51**, 2943.
Lopez, A., and Fradkin, E. (1991). *Physical Review B*, **44**, 5246.
Ni, G. J., and Chen, S. Q. (1995). *Levinson Theorem, Anomaly and the Phase Transition of Vacuum*, Shanghai Scientific and Technical Publishers, Shanghai.
Polyakov, A. M. (1988). *Modern Physics Letters A*, **3**, 325.
Redlich, A. N. (1984a). *Physical Review Letters*, **52**, 18.
Redlich, A. N. (1984b). *Physical Review D*, **29**, 2366.
Semenoff, G. W. (1988). *Physical Review Letters*, **61**, 517.
Yang, C. N. (1989). *Physical Review Letters*, **63**, 2144.
Yang, J. F., and Ni, G. J. (1995). *Physics Letters B*, **343**, 249.